

Determination of the Electromagnetic Lagrangian from a System of Poisson Brackets

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The Lagrangian and Hamiltonian formulations of electromagnetism are reviewed and the Maxwell equations are obtained from the Hamiltonian for a system of many electric charges. It is shown that three of the equations which were obtained from the Hamiltonian, namely the Lorentz force law and two Maxwell equations, can be obtained as well from a set of postulated Poisson brackets. It is shown how the results derived from these brackets can be used to reconstruct the original Lagrangian for the theory aided by some reasoning based on physical concepts.

KEY WORDS: Poisson bracket; Maxwell equations; Lagrangian.

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1. INTRODUCTION

The Maxwell equations provide a type of mathematical summary of several fundamental laws of electromagnetism which originally had their origin in experimental observations. The subsequent development of a non-Abelian extension of the Maxwell theory to the non-Abelian Yang–Mills form and the diverse applications of Yang–Mills to the subatomic realm of high energy physics has probably given further impetus to the study of Maxwell’s theory.

A very novel derivation of a pair of the four Maxwell equations was originally introduced by Feynman, but the exact details of his argument remained unpublished until some of the essential arguments were presented by Dyson (Dyson, 1990). It has also been shown that this procedure can be generalized as well to the case of the dynamics of particles which possess other internal degrees of freedom (Bracken, 1996b). The idea at the heart of these derivations is to postulate a fundamental set of Poisson brackets between the fundamental variables of the system. The basic defining relations of the bracket are then applied to the original system

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of brackets as well as other operations, such as differentiation, to generate further new relations and connections between the variables of the problem (Bracken, 1998). This process is capable of generating some of the Maxwell equations, as we will show.

In this paper, we begin by developing the Hamiltonian formulation of the Maxwell theory, and derive the Maxwell equations in this context. It is shown that the Hamiltonian formalism of the classical system, like the Lagrangian formalism on which it is based, is also invariant under gauge transformations (Kobe, 1981; Kobe and Wen, 1982). Different Hamiltonians can be written so that they all have the same form, often referred to as minimal electromagnetic coupling. Next, the development of the Maxwell equations from a minimal set of defining brackets involving the variables and the basic algebraic properties of the Poisson bracket is reviewed (Bracken, 1998). To this end, a fundamental set of brackets is postulated at the outset in a natural way, and subsequently the algebraic properties of the bracket as well defined analytic operations are applied to obtain the basic Lorentz force law as well as a pair of the Maxwell equations. Based on these initial results, some of the ideas of the inverse problem of the calculus of variations are applied (Bracken, 1999a). To emphasize, the procedure relies on defining a basic set of brackets and using fundamental properties of the bracket to generate new relations, such as the Leibnitz rule and Jacobi identity, regardless of the underlying definition of the bracket. It is shown from these results and with the help of some additional physical motivation at the end that the full Lagrangian for the theory can be reconstructed. In the sense that a Lagrangian theory can be formulated out of a set of elementary results, the complete set of Maxwell's equations can be obtained. The results that are obtained from the Lagrangian by means of the Euler–Lagrange equations, can then be used to define a Hamiltonian for the theory to complete the construction (Kobe and Wen, 1980; Yang, 1976).

It may be asked why this approach is adopted. There are several approaches already that begin this kind of development with commutators (Dyson, 1990). Here we show it is possible to proceed entirely in the classical domain. Of course, the Maxwell equations existed before and independently of quantum mechanics, and nonetheless they are fundamental in generating quantum theories of electromagnetism. The same type of analysis can be carried out on classical theories of gravity and it may prove possible to adopt some of the ideas here to proceed to quantum theories of gravity.

2. LAGRANGIAN AND HAMILTONIAN FORMALISM

The system which is of interest here consists of a collection of nonrelativistic particles which interact with an external electromagnetic field. The Lagrangian for the system is sufficient to be used with the principle of least action to

generate the equations of motion. Moreover, a Lagrangian is required to construct a Hamiltonian for the system.

A system of nonrelativistic particles, each having a charge e_α , mass m_α and a displacement vector $\mathbf{r}_\alpha(t)$ at time t for $\alpha = 1, 2, \dots, N$ in an electric field $\mathbf{E}(\mathbf{x}, t)$ and magnetic field $\mathbf{B}(\mathbf{x}, t)$ can be described by the Lagrangian L , which in Lorentz–Heaviside units takes the form

$$L = \frac{1}{2} \int d^3x (\mathbf{E}^2(\mathbf{x}, t) - \mathbf{B}^2(\mathbf{x}, t)) + \frac{1}{2} \sum_{\alpha=1}^N m_\alpha \dot{\mathbf{r}}_\alpha^2 + \frac{1}{c} \int d^3x (\mathbf{J} \cdot \mathbf{A} - c\rho A_0) - U. \quad (1)$$

For the sake of generality, an arbitrary static external potential energy U can be included in the Lagrangian as well, but it is not essential for what follows. The charge density $\rho(\mathbf{x}, t)$ of the system (1) is defined by

$$\rho(\mathbf{x}, t) = \sum_{\alpha=1}^N e_\alpha \delta(\mathbf{x} - \mathbf{r}_\alpha(t)), \quad (2)$$

and the current density $\mathbf{J}(\mathbf{x}, t)$ associated with the motion of the particle is given by

$$\mathbf{J}(\mathbf{x}, t) = \sum_{\alpha=1}^N e_\alpha \dot{\mathbf{r}}_\alpha \delta(\mathbf{x} - \mathbf{r}_\alpha(t)). \quad (3)$$

Of course, the charge density and current density satisfy the equation of continuity.

The total electromagnetic field is characterized by the vector potential \mathbf{A} and a scalar potential A_0 in an arbitrary gauge. The total electric and magnetic fields are related to the potentials in the following way

$$\mathbf{E}(\mathbf{r}, t) = -\nabla A_0 - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (4)$$

where ∇ is a vector operator and ∇_α is the vector operator corresponding to particle α . The equations of motion can be obtained from Hamilton's principle of least action by varying the action S

$$S = \int_{t_1}^{t_2} dt L.$$

The resulting Euler–Lagrange equations take the general form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_j} = \frac{\partial L}{\partial Q_j} - \sum_i \partial_i \frac{\partial L}{\partial (\partial_i Q_j)}, \quad (5)$$

where the Q_i pertain to any of the physical variables on which L depends. For example, taking Q to be A_0 , variation of the action S with respect to A_0 gives

Gauss's law. This can be obtained as well from the Euler–Lagrange system (5) by identifying Q_i with A_0 and determining the derivatives in (5).

The Hamiltonian formalism for the total system is manifestly gauge invariant, and can be determined by first calculating the canonical momenta. The canonical momentum conjugate to the coordinate \mathbf{r}_α is given using (5) as

$$\mathbf{p}_\alpha = \frac{\partial L}{\partial \dot{\mathbf{r}}_\alpha} = m\dot{\mathbf{r}}_\alpha + \frac{e_\alpha}{c}\mathbf{A}(\mathbf{r}_\alpha, t), \quad (6)$$

The canonical momentum conjugate to the field \mathbf{A} is

$$\mathbf{\Pi} = \frac{\partial L}{\partial \dot{\mathbf{A}}} = -\frac{1}{c}\mathbf{E}. \quad (7)$$

Since the Lagrangian is independent of the quantity \dot{A}_0 , the canonical momentum conjugate to A_0 is $\Pi_0 = 0$. The Hamiltonian for the total system is defined by

$$H = \int d^3x (\mathbf{\Pi} \cdot \dot{\mathbf{A}} + \Pi_0 \dot{A}_0) + \sum_{\alpha=1}^N \mathbf{p}_\alpha \cdot \dot{\mathbf{r}}_\alpha - L. \quad (8)$$

Substituting the canonical momenta $\mathbf{\Pi}$, Π_0 and $\dot{\mathbf{r}}_\alpha$ in terms of \mathbf{p}_α from (6), as well as the Lagrangian L given in (1), we obtain that

$$\begin{aligned} H = & \sum_{\alpha=1}^N \frac{1}{m_\alpha} \mathbf{p}_\alpha \cdot \left(\mathbf{p}_\alpha - \frac{e_\alpha}{c}\mathbf{A} \right) - \frac{1}{2} \int d^3x (\mathbf{E}^2 - \mathbf{B}^2) - \frac{1}{c} \int d^3x (\mathbf{E} \cdot \dot{\mathbf{A}}) \\ & - \frac{1}{2} \sum_{\alpha=1}^N \frac{1}{m_\alpha} \left(\mathbf{p}_\alpha - \frac{e_\alpha}{c}\mathbf{A} \right)^2 + U - \frac{1}{c} \int d^3x (\mathbf{J} \cdot \mathbf{A} - c\rho A_0). \end{aligned} \quad (9)$$

Replacing $\dot{\mathbf{A}}$ in the third term of (9) by solving for (4) $\dot{\mathbf{A}}$, we can write

$$\begin{aligned} H = & \frac{1}{2} \int d^3x (\mathbf{E}^2 + \mathbf{B}^2) + \frac{1}{2} \sum_{\alpha=1}^N \frac{1}{m_\alpha} \left(\mathbf{p}_\alpha - \frac{e_\alpha}{c}\mathbf{A} \right)^2 + U + \int d^3x \mathbf{E} \cdot \nabla A_0 \\ & + \sum_{\alpha=1}^N \frac{e_\alpha}{m_\alpha c} \mathbf{A} \cdot \left(\mathbf{p}_\alpha - \frac{e_\alpha}{c}\mathbf{A} \right) - \frac{1}{c} \int d^3x (\mathbf{J} \cdot \mathbf{A} - c\rho A_0). \end{aligned} \quad (10)$$

Grouping the last three terms in (10) together, this equation can be rewritten using the expressions for ρ and \mathbf{J} given in (2) and (3)

$$\begin{aligned} H = & \frac{1}{2} \int d^3x (\mathbf{E}^2(\mathbf{x}, t) + \mathbf{B}^2(\mathbf{x}, t)) + \sum_{\alpha=1}^N \frac{1}{2m_\alpha} \left(\mathbf{p}_\alpha - \frac{e_\alpha}{c}\mathbf{A}(\mathbf{r}_\alpha, t) \right)^2 \\ & + U + \int d^3x (\rho - \nabla \cdot \mathbf{E}) A_0. \end{aligned} \quad (11)$$

A total derivative term has been dropped to write the Hamiltonian H in the form (11).

From the Hamiltonian (11), Hamilton's equations can be developed. The first of Hamilton's equations for particle α is found by differentiating H with respect to \mathbf{p}_α ,

$$\dot{\mathbf{r}}_\alpha = \frac{\partial H}{\partial \mathbf{p}_\alpha} = \frac{1}{m_\alpha} \left(\mathbf{p}_\alpha - \frac{e_\alpha}{c} \mathbf{A} \right). \quad (12)$$

Using (12) and the vector identity $\nabla_\alpha(\dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha) = 2(\dot{\mathbf{r}}_\alpha \cdot \nabla_\alpha)\dot{\mathbf{r}}_\alpha + 2\dot{\mathbf{r}}_\alpha \times (\nabla_\alpha \times \dot{\mathbf{r}}_\alpha)$, we obtain the second of Hamilton's equations,

$$\begin{aligned} \dot{\mathbf{p}}_\alpha = -\frac{\partial H}{\partial \mathbf{r}_\alpha} &= -\frac{e_\alpha}{c} \dot{\mathbf{r}}_\alpha \cdot \nabla_\alpha \mathbf{A}(\mathbf{r}_\alpha, t) - \frac{e_\alpha}{c} \dot{\mathbf{r}}_\alpha \times (\nabla_\alpha \times \mathbf{A}(\mathbf{r}_\alpha, t)) \\ &+ \frac{\partial U}{\partial \mathbf{r}_\alpha} + e_\alpha \nabla_\alpha A_0(\mathbf{r}_\alpha, t). \end{aligned} \quad (13)$$

Solving for \mathbf{p}_α in (12) and differentiating this with respect to time, we find

$$\dot{\mathbf{p}}_\alpha = m_\alpha \ddot{\mathbf{r}}_\alpha + \frac{e_\alpha}{c} \left(\dot{\mathbf{A}}_\alpha + \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{A}_\alpha}{\partial \mathbf{r}_\alpha} \right). \quad (14)$$

Substituting (14) into (13), and simplifying with Equation (4) yields

$$m \ddot{\mathbf{r}}_\alpha = e_\alpha \mathbf{E}(\mathbf{r}_\alpha, t) + \frac{e_\alpha}{c} \dot{\mathbf{r}}_\alpha \times \mathbf{B}(\mathbf{r}_\alpha, t) - \frac{\partial U}{\partial \mathbf{r}_\alpha}. \quad (15)$$

This is Newton's second law in terms of the Lorentz force.

It remains to complete this process for the field variables and the procedure is the same as for a system with a finite number of degrees of freedom. The functional derivative of H is calculated with respect to $\mathbf{\Pi}$ defined by (7), and this gives the first of Hamilton's equations

$$\dot{\mathbf{A}} = \frac{\delta H}{\delta \mathbf{\Pi}} = -c(\mathbf{E} + \nabla A_0), \quad (16)$$

where the time derivative of \mathbf{A} is a partial derivative.

The second of Hamilton's equations is given by

$$\dot{\mathbf{\Pi}} = -\frac{\delta H}{\delta \mathbf{A}} = -\nabla \times \mathbf{B} + \frac{1}{c} \mathbf{J}, \quad (17)$$

which is the Ampère–Maxwell law,

$$\nabla \times \mathbf{B} = \frac{1}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}. \quad (18)$$

The equation for Π_0 is given by

$$\dot{\Pi}_0 = -\frac{\delta H}{\delta A_0} = -\rho + \nabla \cdot \mathbf{E}. \quad (19)$$

The scalar potential on the right of \mathbf{E} given in (4) is not varied since \mathbf{E} is proportional to $\mathbf{\Pi}$, hence independent. Now, since $\Pi_0 = 0$, Equation (19) implies Gauss's law,

$$\nabla \cdot \mathbf{E} = \rho. \quad (20)$$

Finally, the equation for A_0 is found by differentiating H with respect to Π_0 . Since $\Pi_0 = 0$, this equation is meaningless and does not provide an equation. This procedure has in fact generated all four of Maxwell's equations, although two have not yet been explicitly written down. To obtain the remaining two equations, first take the curl of (16) and use the identity $\nabla \times \nabla A_0 = \mathbf{0}$ and (4),

$$\nabla \times \mathbf{E} = -\nabla \times \nabla A_0 - \frac{1}{c} \frac{\partial}{\partial t} \nabla \times \mathbf{A} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}. \quad (21)$$

This is Faraday's law. The condition that magnetic monopoles do not exist follows by taking the divergence of \mathbf{B} given in (4),

$$\nabla \cdot \mathbf{B} = 0. \quad (22)$$

To summarize, the Maxwell's equations are given by (18), (20), (21) and (22).

3. MAXWELL EQUATIONS AND POISSON BRACKETS

The algebra of classical observables on the manifold M will be denoted by \mathcal{F} . A Poisson structure on a manifold M is a skew-symmetric bilinear map which is denoted $\{, \} : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ such that (i) $(\mathcal{F}, \{, \})$ satisfies the Jacobi identity $\{G, \{H, K\}\} + \{H, \{K, G\}\} + \{K, \{G, H\}\} = 0$, (ii) the map $X_G = \{, G\}$ is a derivation of the associative algebra $\mathcal{F}(M)$ on M (Goldstein, 1950). It satisfies the Leibnitz rule $\{G, HK\} = H\{G, K\} + \{G, H\}K$. A manifold M , which is endowed with a Poisson bracket on $\mathcal{F}(M)$, is called a Poisson manifold. These basic algebraic properties are used on their own without reference to a specific form for the bracket to develop the stated results.

Let the local coordinate variables on the manifold be written in the form $(w^a) = (q^i, v^i)$, where $i = 1, 2, 3$. Indices will be raised and lowered in a trivial way using δ_{ij} , and repeated indices are summed over. Here, the q^i may be interpreted as position coordinates and v^i represent velocity components. The Poisson brackets are postulated in the following way (Bracken, 1996b, 1998),

$$\{q_i, q_j\} = 0, \quad m\{q_i, v_j\} = \delta_{ij}. \quad (23)$$

Any function $H \in \mathcal{F}$ will define a dynamical system on M by the equation

$$\frac{dG}{dt} = \{G, H\}. \quad (24)$$

Let us postulate an $H \in \mathcal{F}$ such that equations of motion can be obtained from (24) as follows,

$$\dot{q}^i = \{q^i, H\} = v^i, \quad m\dot{v}^i = m\{v^i, H\} = F^i. \quad (25)$$

Differentiating the second bracket in (23) with respect to time generates the equation

$$\{\dot{q}_i, v_j\} + \{q_i, \dot{v}_j\} = 0. \quad (26)$$

Multiplying both sides by m and substituting the equations of motion, there results the expression

$$m\{\dot{q}_i, \dot{q}_j\} + \{q_i, F_j\} = 0. \quad (27)$$

Since the bracket is bilinear, this equation can be put into the form

$$\{\{q_i, F_j\}, q_k\} + m\{\{\dot{q}_i, \dot{q}_j\}, q_k\} = 0. \quad (28)$$

Substituting \dot{q}_i, \dot{q}_j and q_k into the Jacobi identity, we have

$$\{\{\dot{q}_i, \dot{q}_j\}, q_k\} + \{\{\dot{q}_j, q_k\}, \dot{q}_i\} + \{\{q_k, \dot{q}_i\}, \dot{q}_j\} = 0. \quad (29)$$

The bracket $\{\dot{q}_j, q_k\}$ is proportional to δ_{jk} by (23), so (29) reduces to the constraint

$$\{\{\dot{q}_i, \dot{q}_j\}, q_k\} = 0. \quad (30)$$

Substituting (30) into (28), we obtain that

$$\{q_k, \{q_i, F_j\}\} = 0. \quad (31)$$

The tensor $\{q_i, F_j\}$ is therefore antisymmetric due to the bracket property. This can be expressed in its dual form by the relation

$$\{q_i, F_j\} = -\frac{e}{mc}\epsilon_{ijk}B_k(\mathbf{r}, t). \quad (32)$$

Substituting (32) into (31), a bracket which contains q_l and B_k can be obtained

$$\{q_l, B_k\} = 0. \quad (33)$$

The postulated relations (23) imply that the vector \mathbf{B} depends only on the position and time of the particle. Equations (33) and (23) imply that F_i is at most linear in the velocities, and so we may write

$$F_i(\mathbf{r}, t) = eE_i(\mathbf{r}, t) + \frac{e}{c}\epsilon_{ijk}v^jB^k(\mathbf{r}, t). \quad (34)$$

This is the Lorentz force law and serves to define the electric field. Using the property of bilinearity and the derivation property, we have

$$\begin{aligned} \left\{ q_i, eE_j + \frac{e}{c}\epsilon_{jak}v_a B_k \right\} &= e\{q_i, E_j\} + \frac{e}{c}\epsilon_{jak}\{q_i, v_a B_k\} = e\{q_i, E_j\} \\ &+ \frac{e}{c}\epsilon_{jak}\{q_i, v_a\}B_k + \frac{e}{c}\epsilon_{jak}v_a\{q_i, B_k\} = e\{q_i, E_j\} \\ &+ \frac{e}{mc}\epsilon_{jik}B_k = e\{q_i, E_j\} - \frac{e}{mc}\epsilon_{ijk}B_k. \end{aligned} \quad (35)$$

Since the result of (35) is the left-hand side of (32), it follows that

$$\{q_i, E_j\} = 0. \quad (36)$$

This bracket implies that the vector \mathbf{E} , as in the case of \mathbf{B} , depends only on the time and position coordinates. Equations (27) and (32) can be combined and will lead to a new equation for B_k in terms of the bracket

$$B^s = \frac{m^2c}{2e}\epsilon^{sij}\{\dot{q}_i, \dot{q}_j\}. \quad (37)$$

Applying the Jacobi identity to the variables \dot{q}_l, \dot{q}_j and \dot{q}_k and then contracting with ϵ^{ljk} , there results

$$\epsilon^{ljk}\{\dot{q}_l, \{\dot{q}_j, \dot{q}_k\}\} = 0.$$

Replacing the bracket $\{\dot{q}_j, \dot{q}_k\}$ in this using (37) generates a new bracket involving \mathbf{B} ,

$$\{\dot{q}_l, \epsilon^{ljk}\{\dot{q}_j, \dot{q}_k\}\} = \frac{2e}{m^2c}\{\dot{q}_l, B_l\} = 0.$$

This implies that $\{\dot{q}_l, B_l\} = 0$, and therefore

$$\nabla \cdot \mathbf{B} = 0. \quad (38)$$

To obtain a second equation, let us begin with the equation for B_s , given in (37). Differentiating both sides with respect to t , we have

$$\frac{\partial B_s}{\partial t} + \frac{\partial B_s}{\partial q_j}\dot{q}^j = \frac{m^2c}{2e}\epsilon^{sij}\{\ddot{q}_i, \dot{q}_j\} + \frac{m^2c}{2e}\epsilon^{sij}\{\dot{q}_i, \ddot{q}_j\} = \frac{m^2c}{e}\{\ddot{q}_i, \dot{q}_i\}. \quad (39)$$

Substituting (34) into the right hand side of (39), dividing out the common factor of e , and using property (ii) of the Poisson bracket, there results

$$\begin{aligned} m\epsilon^{sij} \left\{ E_i + \frac{1}{c}\epsilon_{ial}\dot{q}_a B_l, \dot{q}_i \right\} &= m\epsilon^{sij}\{E_i, \dot{q}_j\} + \frac{m}{c}\{B^s, \dot{q}_j\}\dot{q}_j + \frac{m}{c}\{\dot{q}_j, \dot{q}_j\}B^s \\ &- \frac{m}{c}\{\dot{q}^s, \dot{q}_j\}B_j - \frac{m}{c}\dot{q}^s\{B_j, \dot{q}_j\}. \end{aligned} \quad (40)$$

The second to last term on the right hand side of (40) is zero by symmetry, and upon substitution of the equation $\{q_l, B_l\} = 0$ into (40), this expression reduces to the following

$$\frac{\partial B_s}{c \partial t} + \frac{\partial B_s}{c \partial q_j} \dot{q}^j = -\epsilon_{sji} \frac{\partial E_i}{\partial q_j} + \frac{1}{c} \frac{\partial B_s}{\partial q_j} \dot{q}^j.$$

Simplifying this, the following Maxwell equation is obtained in the usual form

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}. \tag{41}$$

4. RECONSTRUCTION OF THE LAGRANGIAN

The fundamental Poisson brackets have been assumed to have the structure given in (23), in particular, if we combine (23) and (25), the basic relation is of the form

$$m\{q^i, \dot{q}^j\} = \delta^{ij}. \tag{42}$$

Consider the classical equations of motion with all masses set to unity of the form

$$\ddot{q}_i = f_i(q, \dot{q}, t).$$

A nonsingular matrix W_{ij} and a function $L(q, \dot{q}, t)$ are sought such that

$$W_{is}(\ddot{q}^s - F^s) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}. \tag{43}$$

The conditions for the existence of W_{ij} and L are called the Helmholtz conditions (Bracken, 2002). If a Lagrangian L exists, then W_{ij} is given by

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}. \tag{44}$$

From (42), we take W_{ij} to be proportional to δ_{ij} . For the Hessian (44) to be invertible, the Lagrangian must obey

$$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = m\delta_{ij}. \tag{45}$$

Integrating (45), it can be seen that if L exists, it must have the form

$$L = \frac{1}{2} m \dot{q}^i \dot{q}^j \delta_{ij} + \frac{e}{c} \dot{q}^i A_i - e A_0 + C = \frac{1}{2} m \dot{\mathbf{r}}^2 + \frac{e}{c} \dot{\mathbf{r}} \cdot \mathbf{A} - e A_0 + C(\mathbf{E}, \mathbf{B}, \mathbf{A}, A_0). \tag{46}$$

where the quantity C can be regarded as a constant of integration which will be a functional of the fields which were introduced in producing the solution (34) and Equations (38) and (41). Such a term would represent an energy contribution

associated with these fields, and its exact structure can be determined next on physical grounds. The existence of L , however, does follow from the Helmholtz equations, as will be seen. If some physical ideas are now introduced and applied, the Lagrangian (46) can be generalized to the structure given in (1) by using (46) and writing

$$L = \sum_{\alpha=1}^N \frac{1}{2} m_{\alpha} \mathbf{r}_{\alpha}^2 + \frac{1}{c} \int d^3x (\mathbf{J} \cdot \mathbf{A} - c\rho A_0) + \mathcal{C}(\mathbf{E}, \mathbf{B}, \mathbf{A}, A_0), \quad (47)$$

where ρ and \mathbf{J} have been defined in (2) and (3).

We say that the force $\mathbf{F}(t, q^j, \dot{q}^j, \ddot{q}^j)$ is potential if Lagrange's equations of motion

$$\frac{\partial T}{\partial q^i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} = F_i,$$

are variational, that is, if there exists a Lagrange function L such that

$$\frac{\partial T}{\partial q^i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} - F_i = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}.$$

The necessary and sufficient conditions for a force $F^i(t, q^j, \dot{q}^j)$ to be potential can be written

$$\begin{aligned} \frac{\partial F_i}{\partial \dot{q}^j} + \frac{\partial F_j}{\partial \dot{q}^i} &= 0, \\ \frac{\partial F_i}{\partial q^j} - \frac{\partial F_j}{\partial q^i} + \frac{d}{dt} \frac{\partial F_j}{\partial \dot{q}^i} &= 0. \end{aligned} \quad (48)$$

The first of these can be differentiated with respect to \dot{q}^j to give

$$\frac{\partial^2 F_i}{\partial \dot{q}^j \partial \dot{q}^k} = -\frac{\partial^2 F_k}{\partial \dot{q}^i \partial \dot{q}^j} = -\frac{\partial^2 F_j}{\partial \dot{q}^i \partial \dot{q}^k} = -\frac{\partial^2 F_i}{\partial \dot{q}^j \partial \dot{q}^k},$$

therefore,

$$\frac{\partial^2 F_i}{\partial \dot{q}^j \partial \dot{q}^k} = 0.$$

This equation can now be integrated to give the result $F_i = a_{ij} \dot{q}^j + b_i$, where a_{ij} and b_i are functions of (t, q^k) . Substituting F_i into the pair (48), we obtain a set of three conditions on the a_{ij} and b_i as follows

$$\begin{aligned} a_{ij} &= -a_{ji}, \\ \frac{\partial a_{is}}{\partial q^j} + \frac{\partial a_{sj}}{\partial q^i} + \frac{\partial a_{ji}}{\partial q^s} &= 0, \\ \frac{\partial b_i}{\partial q^j} - \frac{\partial b_j}{\partial q^i} &= \frac{\partial a_{ij}}{\partial t}. \end{aligned} \quad (49)$$

By setting $a_{ij} = -(e/c)\epsilon_{ijk}B^k$ and $eE^i = \delta^{ij}b_j$, we obtain the Lorentz force law (34) after reversing the direction of the velocity vector, such that \mathbf{E} and \mathbf{B} satisfy,

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}.$$

These results are consistent with the development from the Poisson bracket point of view. It is interesting to note that if we make the opposite selection $b_i = \delta_{ij}B^j$ and $a_{ij} = \epsilon_{ijk}E^k$, albeit for which there is no known force law, the remaining two sourceless Maxwell equations appear. Of course, the Maxwell equations are known to be symmetric under the transformations $\mathbf{E} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow -\mathbf{E}$.

To determine the quantity \mathcal{C} in (47), we have to be allowed to reason from a physical point of view. This functional will be a scalar formed from the fields \mathbf{E} and \mathbf{B} , which will serve as a Lagrangian for these fields. To this end, we construct a functional of the form

$$L_{em} = \int d^3x (\alpha \mathbf{E}^2(\mathbf{x}, t) + \beta \mathbf{B}^2(\mathbf{x}, t) + \gamma \mathbf{E}(\mathbf{x}, t) \cdot \mathbf{B}(\mathbf{x}, t)), \quad (50)$$

and identify L_{em} with the integration constant \mathcal{C} . This reflects the fact that the fields involved will have an energy density associated with them in the absence of particles. Adding terms such as \mathbf{A}^2 and A_0^2 to this integrand would destroy the form invariance of the total Lagrangian under the addition of a total time derivative.

Since \mathbf{E} is a vector and \mathbf{B} is a pseudovector, this will be invariant with respect to parity only if $\gamma = 0$. The constants α and β can be taken so that other equations and quantities obtained from L will have their standard forms, in this case, $\alpha = \beta = 1/2$. Now, from the Lagrangian and the principle of least action, we can immediately determine the remaining two Maxwell equations. If the action is varied with respect to A_0 , Gauss's law results and if we vary with respect to \mathbf{A} , then the Ampère–Maxwell law results as before.

To summarize, it has been shown that a set of basic Poisson brackets leads to some of the basic structures in electromagnetism, in particular, the Lorentz force law. This can then be used in turn to develop a Lagrangian which must exist on account of the Helmholtz conditions. It is also quite interesting that without any direct appeal to Lorentz invariance, a system of equations results which can be shown to be relativistically invariant under Lorentz transformations (Berard, 1999; Berard and Mohrbach, 2000).

It can be seen that the Maxwell equations are partitioned into two groups, and this is clearly indicated in the details of the reconstruction. Both Faraday's law (21) and (22) are kinematical laws. They follow from the relationship between the actual fields and the charged particles. It is not surprising then that these two equations are generated by the set of Poisson brackets (van Holten, 1991). On the other hand, the Ampère–Maxwell law (18) and Gauss's law (20) are dynamical equations. It would be of interest to know whether other types of equations of

physical interest, for example to the realm of gravity (Tanimura, 1992), can be developed along similar lines as described here.

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